Adjustable Robust Optimization with discrete uncertainty

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Problem definition

MILP with uncertain matrix

$$\text{minimize } \boldsymbol{c}^T \boldsymbol{x} + \boldsymbol{d}^T \boldsymbol{y} \tag{1}$$

$$\text{subject to } \boldsymbol{T}\boldsymbol{x} + \boldsymbol{H}\boldsymbol{y} \leq \boldsymbol{f} \tag{2}$$

$$\boldsymbol{x} \in X, \boldsymbol{y} \in Y \tag{3}$$

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- We can represent $\mathcal{H}$ thanks to binary parameters $\xi_{ij} \in \{0, 1\}$

$$h_{ij} = \underline{h}_{ij} + (\bar{h}_{ij} - \underline{h}_{ij})\xi_{ij} \qquad \boldsymbol{\xi} \in \Xi \tag{4}$$

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- For $\hat{\boldsymbol{x}} \in X$ and appropriate $\hat{\boldsymbol{\xi}} \in \Xi$, $Y(\boldsymbol{x}, \hat{\boldsymbol{H}})$ is encoded as

$$Y(\hat{\boldsymbol{x}}, \hat{\boldsymbol{\xi}}) = \left\{ \boldsymbol{y} \in Y : \sum_{j=1}^{n_Y} \left(\underline{h}_{ij} + (\bar{h}_{ij} - \underline{h}_{ij})\,\hat{\xi}_{ij} \right) y_j \leq f_i - \sum_{j=1}^{n_X} t_{ij}\,\hat{x}_j \quad i = 1, ..., m_Y \right\} \tag{5}$$

An adjustable robust approach

$$\min_{\boldsymbol{x}\in X}\left\{\boldsymbol{c}^T\boldsymbol{x}+\max_{\boldsymbol{\xi}\in\Xi}\min_{\boldsymbol{y}\in Y(\boldsymbol{x},\boldsymbol{\xi})}\boldsymbol{d}^T\boldsymbol{y}\right\} \tag{6}$$

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Here and now	*Uncertainty*	*Wait and see*
Make decision $\boldsymbol{x}\in X$ based on *a priori* knowledge $\boldsymbol{\xi}\in\Xi$	Observe the real outcome $\overline{\boldsymbol{\xi}}$ of $\tilde{\boldsymbol{\xi}}$	Make recourse decision $\boldsymbol{y}\in Y(\boldsymbol{x},\overline{\boldsymbol{\xi}})$ based on *a posteriori* knowledge $\overline{\boldsymbol{\xi}}$

→ time

Aim of this work

- Some bad news...
  1. These problems include Σ_2^P-hard
     - ⋆ Includes Knapsack Interdiction Problem
  2. Most of the literature considers $\mathrm{conv}\,(\Xi)$ instead of Ξ
  3. Mixed-integer second-stage (even when Ξ is convex) are very hard to deal with
     - ⋆ Dual approaches are no longer feasible
     - ⋆ column-and-constraint generation MP hard to solve, bilevel problem with integer follower for separation...

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- Some encouraging results
  1. Efficient approaches for the sepcial case of objective uncertainty and convex uncertainty
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- Our contribution
  1. **Constraint uncertainty = Objective uncertainty** for binary Ξ
  2. We can then apply the results from Kämmerling and Kurtz (2020)

Reformulation for ARO with binary uncertainty

Step 1/5: linearization

- Remember the second stage

$$Y(\hat{\boldsymbol{x}}, \hat{\boldsymbol{\xi}}) = \left\{ \boldsymbol{y} \in Y : \sum_{j=1}^{n_Y} \left(\underline{h}_{ij} y_j + (\bar{h}_{ij} - \underline{h}_{ij}) \hat{\xi}_{ij} y_j \right) \leq f_i - \sum_{j=1}^{n_X} t_{ij} \hat{x}_j \quad i = 1, ..., m_Y \right\} \tag{7}$$

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- Introduce $z_{ij} = \xi_{ij} y_j$

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- Linearize

$$z_{ij} \le u_j \xi_{ij} \qquad i = 1, ..., m_Y, j = 1, ..., n_Y \tag{9}$$

$$z_{ij} \le y_j \qquad i = 1, ..., m_Y, j = 1, ..., n_Y \tag{10}$$

$$z_{ij} \ge y_j - (1 - \xi_{ij}) u_j \qquad i = 1, ..., m_Y, j = 1, ..., n_Y \tag{11}$$

$$z_{ij} \ge 0 \qquad i = 1, ..., m_Y, j = 1, ..., n_Y \tag{12}$$

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- Thus, the second stage is

$$\min_{\boldsymbol{y}\in Y(\boldsymbol{x},\boldsymbol{\xi})} \boldsymbol{d}^T\boldsymbol{y} = \min_{(\boldsymbol{y},\boldsymbol{z})\in Z(\boldsymbol{x},\boldsymbol{\xi})} \boldsymbol{d}^T\boldsymbol{y} \tag{13}$$

with

$$Z(\boldsymbol{x},\boldsymbol{\xi}) = \left\{ (\boldsymbol{y},\boldsymbol{z}) : \begin{array}{ll} \sum_{j=1}^{n_Y} \left(\underline{h}_{ij} y_j + (\bar{h}_{ij} - \underline{h}_{ij}) z_{ij} \right) \le f_i - \sum_{j=1}^{n_X} t_{ij} x_j & i \in [m_Y] \\ z_{ij} \le y_j & i \in [m_Y], j \in [n_Y] \\ z_{ij} \ge 0 & i \in [m_Y], j \in [n_Y] \\ z_{ij} \ge y_j - (1 - \xi_{ij}) u_j & i \in [m_Y], j \in [n_Y] \\ \boldsymbol{y} \in Y & \end{array} \right\} \tag{14}$$

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- Define $Z'(\boldsymbol{x})$ such that

$$Z(\boldsymbol{x},\boldsymbol{\xi}) = Z'(\boldsymbol{x}) \cap \{(\boldsymbol{y},\boldsymbol{z}) : z_{ij} \ge y_j - (1-\xi_{ij})u_j \quad i \in [m_Y], j \in [n_Y]\} \tag{15}$$

Polyhedral analysis

Theorem (Arslan and Detienne (2021), Li and Grossman (2018))

Let $Y \subseteq \Pi_{j=1}^n [l_j, u_j]$ *and let* $L(\boldsymbol{x})$ *be defined, for* $\boldsymbol{x} \in \{0,1\}^n$ *as follows,*

$$L(\boldsymbol{x}) = \left\{ \boldsymbol{y} \in \mathbb{R}_+^n : \forall j \in \{1, ..., n\}, \begin{cases} x_j = 1 \Rightarrow y_j \in [\alpha_j^1, \beta_j^1] \\ x_j = 0 \Rightarrow y_j \in [\alpha_j^0, \beta_j^0] \end{cases} \right\} \tag{16}$$

with $\alpha_j^0, \alpha_j^1, \beta_j^0, \beta_j^1 \in \{l_j, u_j\}$. *Then, the following equality holds,*

$$\forall \boldsymbol{x} \in \{0,1\}^n, \quad \mathrm{conv}(Y \cap L(\boldsymbol{x})) = \mathrm{conv}(Y) \cap L(\boldsymbol{x}) \tag{17}$$

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Corollary

We had

$$Z(\boldsymbol{x}, \boldsymbol{\xi}) = Z'(\boldsymbol{x}) \cap \{(\boldsymbol{y}, \boldsymbol{z}) : z_{ij} \geq y_j - (1 - \xi_{ij})u_j \quad i \in [m_Y], j \in [n_Y]\} \quad (18)$$

We also have

$$\mathrm{conv}\left(Z(\boldsymbol{x}, \boldsymbol{\xi})\right) = \mathrm{conv}\left(Z'(\boldsymbol{x})\right) \cap \{(\boldsymbol{y}, \boldsymbol{z}) : z_{ij} \geq y_j - (1 - \xi_{ij})u_j \quad i \in [m_Y], j \in [n_Y]\} \quad (19)$$

Step 2/5: convexify

- By linearity of the objective function

$$\min_{\boldsymbol{y}\in Y(\boldsymbol{x},\boldsymbol{\xi})} \boldsymbol{d}^T\boldsymbol{y} = \min_{(\boldsymbol{y},\boldsymbol{z})\in\operatorname{conv}(Z(\boldsymbol{x},\boldsymbol{\xi}))} \boldsymbol{d}^T\boldsymbol{y} \tag{20}$$

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- Thus,

$$\min_{\boldsymbol{y}\in Y(\boldsymbol{x},\boldsymbol{\xi})} \boldsymbol{d}^T\boldsymbol{y} = \min_{\substack{(\boldsymbol{y},\boldsymbol{z})\in\operatorname{conv}(Z'(\boldsymbol{x}))\\ z_{ij}\geq y_j-(1-\xi_{ij})u_j}} \boldsymbol{d}^T\boldsymbol{y} \tag{23}$$

Step 3/5: dualize

- The second stage problem is now an LP!

$$\text{minimize } \boldsymbol{d}^T \boldsymbol{y} \tag{24}$$

$$\text{subject to } (\boldsymbol{y}, \boldsymbol{z}) \in \text{conv}\left(Z'(\boldsymbol{x})\right) \tag{25}$$

$$z_{ij} \geq y_j - (1 - \xi_{ij})u_j \qquad (\lambda_{ij} \leq 0) \tag{26}$$

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- It is equal to its dual

$$= \max_{\boldsymbol{\lambda} \leq 0} \min_{(\boldsymbol{y},\boldsymbol{z}) \in Z'(\boldsymbol{x})} \left\{ \sum_{j=1}^{n_Y} d_j y_j + \sum_{i=1}^{n_Y} \sum_{j=1}^{n_Y} \lambda_{ij} ((1 - \xi_{ij}) u_j + z_{ij} - y_j) \right\} \tag{27}$$

Step 4/5: re-arrange

- For a given $\boldsymbol{\xi} \in \Xi$, let us rearrange the terms

$$\max_{\boldsymbol{\lambda} \leq 0} \min_{(\boldsymbol{y},\boldsymbol{z}) \in Z'(\boldsymbol{x})} \left\{ \sum_{j=1}^{n_Y} d_j y_j + \sum_{i=1}^{n_Y} \left(\sum_{j:\xi_{ij}=0} \lambda_{ij}(u_j + z_{ij} - y_j) + \sum_{j:\xi_{ij}=1} \lambda_{ij}(z_{ij} - y_j) \right) \right\} \tag{28}$$

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- By inspection, since $u_j + z_{ij} - y_j \geq 0$ and $\lambda_{ij} \leq 0$, $\xi_{ij} = 0 \Rightarrow \lambda^*_{ij} = 0$
- Thus, we can write

$$= \max_{\boldsymbol{\lambda} \leq 0} \min_{(\boldsymbol{y},\boldsymbol{z}) \in Z'(\boldsymbol{x})} \sum_{j=1}^{n_Y} \left(d_j y_j + \sum_{i=1}^{m_Y} \lambda_{ij} \xi_{ij} (z_{ij} - y_j) \right) \tag{29}$$

Step 5/5: dual fixation

- We obtain

$$\min_{\boldsymbol{x}\in X} \left\{ \sum_{j=1}^{n_X} c_j x_j + \max_{\boldsymbol{\xi}\in\Xi, \boldsymbol{\lambda}\leq 0} \min_{(\boldsymbol{y},\boldsymbol{z})\in Z'(\boldsymbol{x})} \sum_{j=1}^{n_Y} \left(d_j y_j + \sum_{i=1}^{m_Y} \lambda_{ij}\xi_{ij}(z_{ij} - y_j) \right) \right\} \tag{30}$$

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- **We can replace λ_{ij} by a sufficiently large value $\underline{\lambda}_{ij}$!**

  (i.e., bounds on $\lambda^*_{ij}(\boldsymbol{\xi})$ for all $\boldsymbol{\xi}$)

- For the special case of downward monotone second stage, $\underline{\lambda}_{ij} = d_j$ is large enough!

  (i.e., $d_j \le 0$ and $\underline{h}_{ij} \ge 0$)

Summary

- We have shown that the following problem

$$\min_{\boldsymbol{x}\in X} \left\{ \boldsymbol{c}^T\boldsymbol{x} + \max_{\boldsymbol{\xi}\in\Xi} \min_{\boldsymbol{y}\in Y(\boldsymbol{x},\boldsymbol{\xi})} \boldsymbol{d}^T\boldsymbol{y} \right\} \tag{31}$$

is equivalently solved by the following one

$$\min_{\boldsymbol{x}\in X} \left\{ \sum_{j=1}^{n_X} c_j x_j + \max_{\boldsymbol{\xi}\in\Xi} \min_{(\boldsymbol{y},\boldsymbol{z})\in Z'(\boldsymbol{x})} \sum_{j=1}^{n_Y} \left(d_j y_j + \sum_{i=1}^{m_Y} \underline{\lambda}_{ij}\xi_{ij}(z_{ij} - y_j) \right) \right\} \tag{32}$$

- We may now use the algorithmic approach of Kämmerling and Kurtz (2020)

Application to a Facility Location Problem

Capacity Facility Location Problem (CFLP)

- Given a set of sites (in green) and a set of clients (in red), where should we open a facility in order to *efficiently* serve our clients?

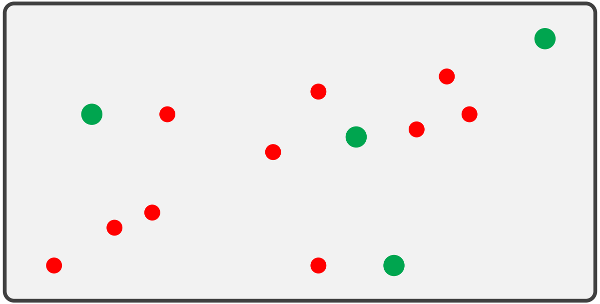

Figure: Example of CFLP instance

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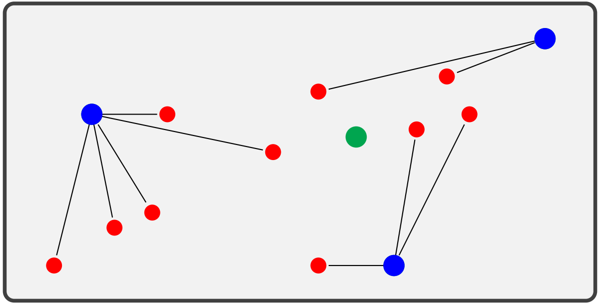

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  - d_j the demand of client j
  - p_j the unitary profit for serving client j

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- Let V_1 be a set of sites and, for all $i \in V_1$, define
  - q_i the capacity of site i
  - f_i the opening cost of site i
- Let V_2 be a set of clients and, for all $j \in V_2$, define
  - d_j the demand of client j
  - p_j the unitary profit for serving client j
- For every connection $(i, j) \in V_1 \times V_2$, define
  - t_{ij} the unitary transportation cost from i to j

Uncertainty

- Demands are uncertain

$$d_j = \bar{d}_j \pm \tilde{d}_j \tag{33}$$

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- We introduce Ξ such that

$$(\boldsymbol{l}, \boldsymbol{h}) \in \Xi \Leftrightarrow \begin{cases} \hat{d}_j = \bar{d}_j - \tilde{d}_j & \text{if } l_j = 1 \text{ and } h_j = 0 \\ \hat{d}_j = \bar{d}_j + \tilde{d}_j & \text{if } l_j = 0 \text{ and } h_j = 1 \\ \hat{d}_j = \bar{d}_j & \text{if } l_j = 0 \text{ and } h_j = 0 \end{cases} \tag{34}$$

and at most Γ clients change their demands

Model

- **Here-and-now decisions**: $X = \{0,1\}^{|V_1|}$, opening facilities

$$x_i = 1 \Leftrightarrow \text{ site } i \text{ is opened} \tag{35}$$

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Model

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- **Uncertainty**: $\hat{d}_j = \bar{d}_j - l_j\tilde{d}_j + h_j\tilde{d}_j$
- **Second-stage decisions**: Let $\hat{\boldsymbol{x}} \in X$ and $(\hat{\boldsymbol{l}}, \hat{\boldsymbol{h}}) \in \Xi$,

$$\sum_{i\in V_1} f_i x_i + \text{minimize} \sum_{(i,j)\in V_1\times V_2} t_{ij}s_{ij} - \sum_{j\in V_1} p_j(\bar{d}_j - \hat{l}_j\tilde{d}_j + \hat{h}_j\tilde{d}_j)y_j \tag{36}$$

$$\sum_{i\in V_1} s_{ij} \geq y_j(\bar{d}_j - \hat{l}_j\tilde{d}_j + \hat{h}_j\tilde{d}_j) \quad \forall j \in V_2 \tag{37}$$

$$\sum_{j\in V_2} s_{ij} \leq q_i\hat{x}_i \tag{38}$$

$$s_{ij} \geq 0, y_j \in \{0,1\} \quad (i,j) \in V_1 \times V_2 \tag{39}$$

Reformulation

$$\min_{\boldsymbol{x}\in X} \max_{(\boldsymbol{l},\boldsymbol{h})\in\Xi} \min_{(\boldsymbol{y},\boldsymbol{s},\boldsymbol{z}^l,\boldsymbol{z}^h)\in Z'(\boldsymbol{x})} \Pi(\boldsymbol{y},\boldsymbol{s},\boldsymbol{z}^l,\boldsymbol{z}^h,\boldsymbol{l},\boldsymbol{h}) \tag{40}$$

Reformulation

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where

$$\Pi(\boldsymbol{y},\boldsymbol{s},\boldsymbol{z}^l,\boldsymbol{z}^h,\boldsymbol{l},\boldsymbol{h}) = \sum_{v\in V_2}\left(\sum_{u\in V_1} t_{uv}s_{uv} - p_v(\bar{d}_v - \tilde{d}_v l_v + \tilde{d}_v h_v)y_v + \underline{\lambda}_v^h h_v(y_v - z_v^h) + (1-l_v)\underline{\lambda}_v^l z_v^l\right) \tag{41}$$

with, for all $v \in V_2$, $\underline{\lambda}_v^l = p_v(\bar{d}_v - \tilde{d}_v)$, $\underline{\lambda}_v^h = p_v(\bar{d}_v + \tilde{d}_v)$

Reformulation

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with, for all $v\in V_2$, $\underline{\lambda}_v^l = p_v(\bar{d}_v - \tilde{d}_v)$, $\underline{\lambda}_v^h = p_v(\bar{d}_v + \tilde{d}_v)$ and,

$$Z'(\boldsymbol{x}) = \left\{ (\boldsymbol{y},\boldsymbol{s},\boldsymbol{z}^l,\boldsymbol{z}^h) : \begin{array}{l} \boldsymbol{y}\in\{0,1\}^{|V_2|}, \boldsymbol{s}\in\mathbb{R}_+^{|V_1|\times|V_2|}, \boldsymbol{z}^l\in\{0,1\}^{|V_2|}, \boldsymbol{z}^h\in\{0,1\}_+^{|V_2|} \\ \sum_{u\in V_1} s_{uv} \ge \bar{d}_v y_v - \tilde{d}_v z_v^l + \tilde{d}_v z_v^h \quad \forall v\in V_2 \\ z_v^l \le y_v \quad \forall v\in V_2 \\ z_v^h \le y_v \quad \forall v\in V_2 \end{array} \right\} \tag{42}$$

Experimental results

- AMD 3960 running at 3.8 GHz
- 3600s time limit
- C++17 using IBM CPLEX version 12.10 to solve every sub-problem.

			$\mu = 1.5$				$\mu = 2.0$				All			
$\|V_1\|$	$\|V_2\|$	Γ	opt.	time	nodes	cuts	opt.	time	nodes	cuts	opt.	time	nodes	cuts
6	12	2	16	0.9	2.4	80.3	16	0.8	2.3	65.3	32	0.9	2.3	72.8
		4	16	20.6	2.5	380.6	16	29.5	2.1	420.9	32	25.1	2.3	400.8
		6	16	117.9	3.1	825.1	15	107.0	1.9	633.5	31	112.6	2.5	732.4
8	16	2	16	3.5	2.4	155.5	16	2.8	2.1	136.6	32	3.2	2.3	146.1
		4	15	367.4	2.5	1338.5	15	173.9	2.2	947.6	30	270.6	2.3	1143.1
		6	5	143.7	1.4	709.2	11	845.5	1.7	1682.8	16	626.1	1.6	1378.6
10	20	2	16	9.4	3.3	282.2	16	6.4	2.0	179.5	32	7.9	2.6	230.8
		4	11	752.1	3.2	1990.0	14	549.3	1.7	1285.0	25	638.5	2.4	1595.2
		6	3	1150.2	1.0	1812.7	7	1123.1	1.0	1318.0	10	1131.3	1.0	1466.4
12	24	2	16	18.6	2.3	335.9	16	15.7	1.9	288.9	32	17.1	2.1	312.4
		4	9	1277.1	1.9	2106.4	5	797.1	1.4	1616.2	14	1105.7	1.7	1931.4
		6	2	708.7	1.0	1509.0	1	2173.8	1.0	1926.0	3	1197.1	1.0	1648.0

Conclusion

- We have proposed a generic reformulation technique for ARO with binary uncertainty
- The reformulation makes the second-stage independent of the uncertain parameters
- We have applied our approach to a Facility Location Problem using the existing literature