Reformulation for Adjustable Robust Optimization with discrete uncertainty

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Capacitated Facility Location Problem...

- **Objective**: decide where to open facilities
- **Constraints**:
  - sites (rectangles) have limited capacities q_i
  - clients (circles) must be served entirely or not at all
- **Minimize**:
  "Opening costs" + "Transportation costs" − "Profit"

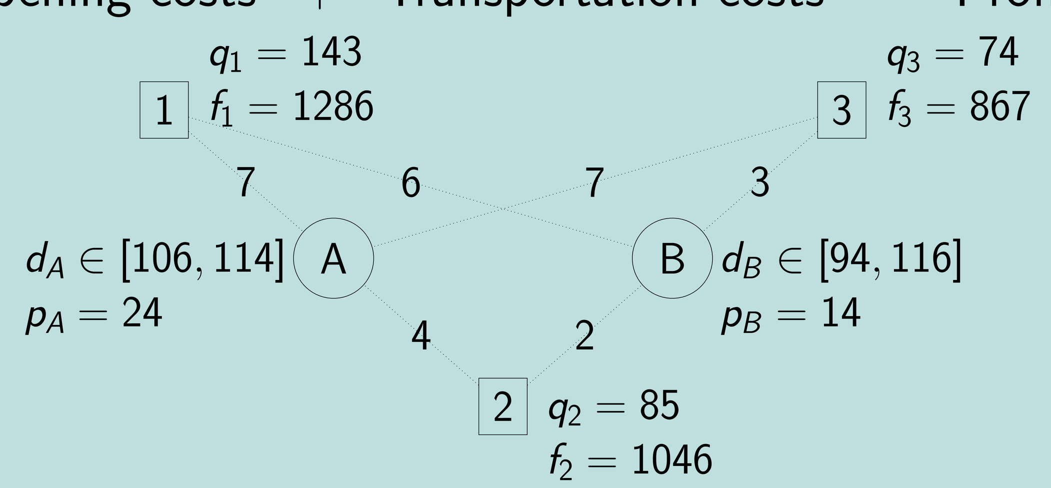


Figure: CFLP instance

...with Uncertain Demand

- **Assumption**: at most Γ clients change their demand

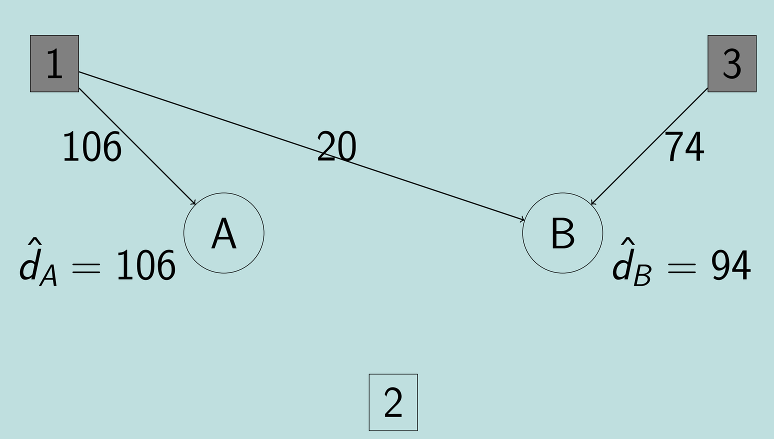


Figure: No client change demand ($\Gamma = 0$)

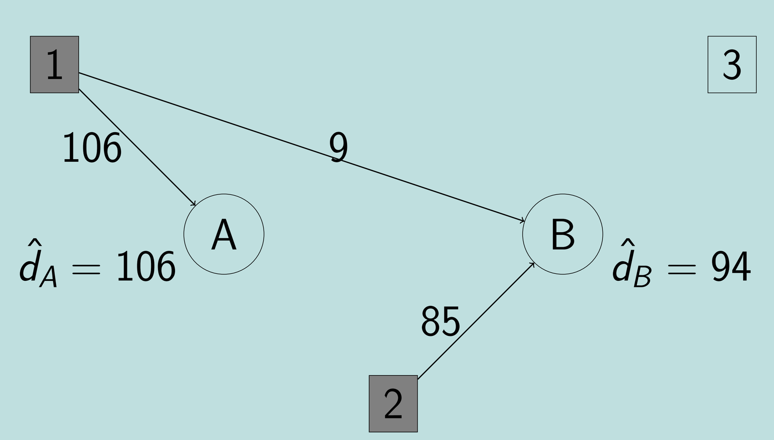


Figure: At most one client change demand ($\Gamma = 1$)

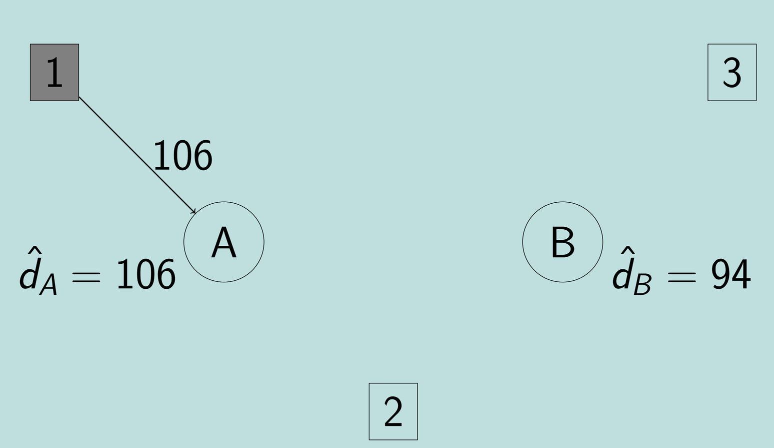
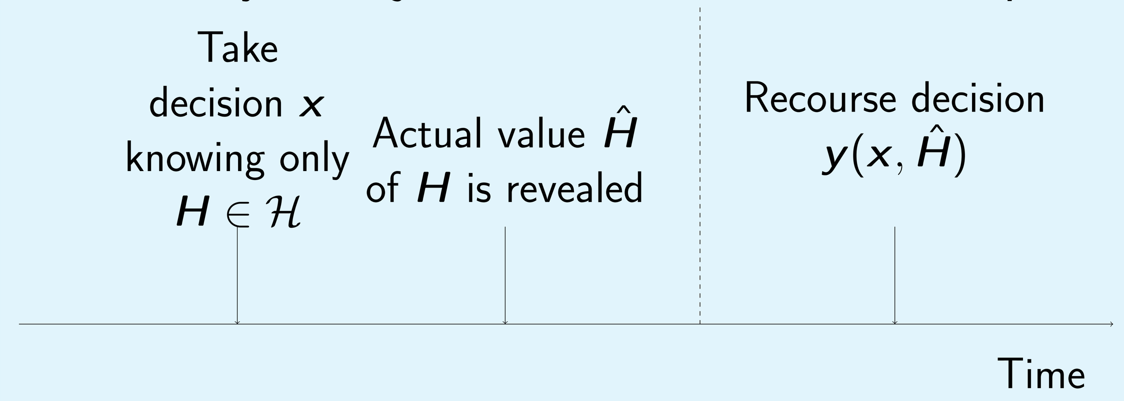


Figure: At most two clients change demand ($\Gamma = 2$)

In gray, sites in which it is optimal to open a facility.

Adjustable Robust Optimization

- Decision $x \in X$ must be taken *here and now*
- Uncertain parameters $H \in \mathcal{H} = \{\hat{H}^1, ..., \hat{H}^L\}$
- Possibility to adjust later, in a *wait-and-see* phase



Assumptions

- (MILP first stage) $X \subseteq \mathbb{R}^{n_X}$
- (Discrete uncertainty) For all coefficient h_{ij},
  $$h_{ij} = \underline{h}_{ij} \text{ or } h_{ij} = \bar{h}_{ij}$$
  (only two values is wlog)
- (MILP second stage) $\forall x \in X$, $\forall \hat{H} \in \mathcal{H}$,
  $$Y(x, \hat{H}) = \left\{ y \in Y : Tx + \hat{H}y \leq f \right\}$$
  with $Y \subseteq \mathbb{R}^{n_Y}$

Main Result: "constraint uncertainty = objective uncertainty"

$$\min_{x \in X} \max_{\xi \in \Xi} \min_{y \in Y(x,\xi)} f(x, y) = \min_{x \in X} \max_{\xi \in \Xi} \min_{(y,z) \in \widetilde{Y}(x)} \widetilde{f}(x, y, z, \xi)$$

A six-step reformualtion

(1) Binary encoding

- Introduce $\xi_{ij} \in \{0, 1\}$ such that $\xi_{ij} = 1$ iff $h_{ij} = \bar{h}_{ij}$
  $$\sum_{j=1}^{n_X} t_{ij}\, \hat{x}_j + \sum_{j=1}^{n_Y} \left(\underline{h}_{ij} y_j + (\bar{h}_{ij} - \underline{h}_{ij})\, \hat{\xi}_{ij}\, y_j \right) \leq f_i$$

(2) Extended space

- Linearize! Introduce $z_{ij} = \xi_{ij} y_j$
  $$z_{ij} \leq y_j \qquad z_{ij} \geq y_j - (1 - \xi_{ij}) u_j \qquad z_{ij} \geq 0$$
- "$z_{ij} \leq u_j \xi_{ij}$" can be omitted by optimality

(3) Convexification

- Define $\widetilde{Y}(x)$ as the set of (y, z) with $y \in Y$ and,
  $$\begin{cases} \sum_{j=1}^{n_X} t_{ij}\, \hat{x}_j + \sum_{j=1}^{n_Y} \left(\underline{h}_{ij} y_j + (\bar{h}_{ij} - \underline{h}_{ij})\, z_{ij} \right) \leq f_i \\ 0 \leq z_{ij} \leq y_j \end{cases}$$
- Then,
  $$\min_{y \in Y(x,\xi)} d^T y = \min_{\substack{(y,z) \in \mathrm{conv}\left(\widetilde{Y}(x)\right) \\ (1-\xi_{ij})u_j \geq y_j - z_{ij}}} d^T y$$

(4) Dualization

- By duality, the second stage is equivalent to
  $$\max_{\lambda \leq 0} \min_{(y,z) \in \widetilde{Y}(x)} \left\{ \sum_{j=1}^{n_Y} d_j y_j + \sum_{i=1}^{m_Y} \sum_{j=1}^{n_Y} \lambda_{ij} \left((1 - \xi_{ij}) u_j + z_{ij} - y_j \right) \right\}$$

(5) Re-writing

- Re-arrange the terms
  $$\sum_{j=1}^{n_Y} d_j y_j + \sum_{i=1}^{n_Y} \left(\sum_{j:\xi_{ij}=0} \lambda_{ij}(u_j + z_{ij} - y_j) + \sum_{j:\xi_{ij}=1} \lambda_{ij}(z_{ij} - y_j) \right)$$
- "$(u_j + z_{ij} - y_j)$" is always non-negative
- In turn, we can re-write the second-stage problem,
  $$\max_{\lambda \leq 0} \min_{(y,z) \in Z'(x)} \sum_{j=1}^{n_Y} \left(d_j y_j + \sum_{i=1}^{m_Y} \lambda_{ij} \xi_{ij} (z_{ij} - y_j) \right)$$

(6) Fixation

- Variables $\lambda_{ij} \leq 0$ can be replaced by a sufficiently small value $\underline{\lambda}_{ij}$ (big-M approach)
- For downward monotone second stage, $\underline{\lambda}_{ij} = d_j$
- *Examples*: MKP, CFLP, $1|r_j|\sum w_j U_j$, ...

A cut-generation LB problem

- Note that $\min_x \max_\xi \min_{(y,z)} \geq \max_\xi \min_{(x,y,z)}$
- Let $(x^1, y^1, z^1), ..., (x^H, y^H, z^H) \in W$ with
  $$W = \{(x, y, z) : x \in X, (y, z) \in \widetilde{Y}(x)\}$$
- The following problem is lower bounding
  $$\begin{aligned} &\max \theta \\ &\text{s.t. } \theta \leq \widetilde{f}(x, y, z, \xi) \quad h = 1, ..., H \\ &\quad \xi \in \Xi \end{aligned}$$
  with $\widetilde{f}(x, y, z, \xi) = d^T y + \sum_{i=1}^{m_Y} \lambda_{ij} \xi_{ij} (z_{ij} - y_j)$
- We can solve this by cut generation!

Asymptotic convergent B&B

- Branch on $\bar{x} = \frac{1}{H} \sum_{h=1}^{H} x^h$
- Finite convergence if $X \subseteq \{0, 1\}^{n_X}$
- Spatial branching on continuous variables
- Generalizes the approach from [1]

Experimental results (CFLP)

- μ is the ratio "total capacity over demand"

			$\mu = 1.5$		$\mu = 2.0$	
sites	clients	Γ	opt	time	opt	time
6	12	2	16	0.9	16	0.8
		4	16	20.6	16	29.5
		6	16	117.9	15	107.0
8	16	2	16	3.5	16	2.8
		4	15	367.4	15	173.9
		6	5	143.7	11	845.5
10	20	2	16	9.4	16	6.4
		4	11	752.1	14	549.3
		6	3	1150.2	7	1123.1
12	24	2	16	18.6	16	15.7
		4	9	1277.1	5	797.1
		6	2	708.7	1	2173.8

Table: Computational results on CFLP instances

References

[1] N. Kämmerling and J. Kurtz.
Oracle-based algorithms for binary two-stage robust optimization.
Computational Optimization and Applications, 77(2):539–569, 2020.

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