Computing Counterfactual Explanations of Linear Problems

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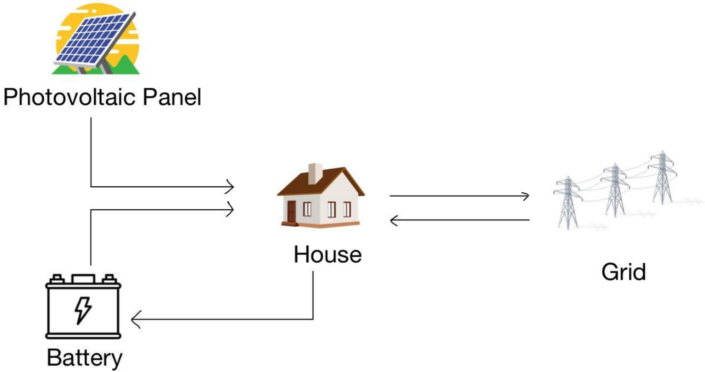
Understanding Energy Model Decisions



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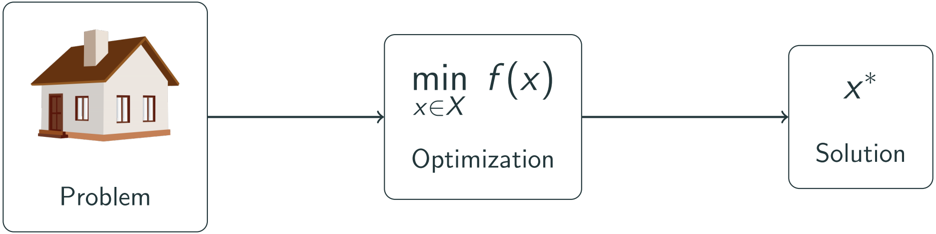


Understanding Energy Model Decisions

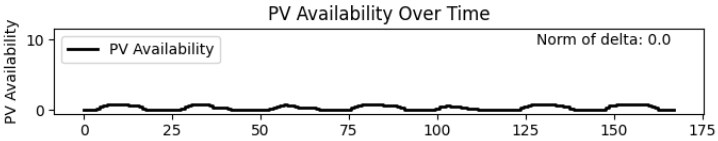
Problem

$$\min_{x \in X} \; f(x)$$

Optimization

$$x^*$$

Solution



Input Data

Outline

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The Underlying Optimization Problem

We consider the linear optimization problem

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Goal: Find an x so that an optimal point y^* is in the desired space Y

Problem Formulation

$$
\begin{aligned}
\inf_{x,y} \quad & f(x) \\
\text{s.t.} \quad & x \in X \\
& y \in Y \\
& y \in \arg\min_{\bar{y}} \left\{ f^\top \bar{y} \;:\; D(x)\bar{y} \geq b \right\}
\end{aligned}
$$

Strong-Duality Reformulation

Bilevel Constraint

$$y \in \arg\min_{\bar{y}} \left\{ f^\top \bar{y} \;:\; D(x)\bar{y} \geq b \right\}$$

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Single-Level Constraint

There exists λ s.t.

$$D(x)y \geq b$$
$$D(x)^\top \lambda = f, \quad \lambda \geq 0$$
$$f^\top y \leq b^\top \lambda$$

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A General Optimization Problem

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\begin{aligned}
\min_{x, y} \quad & F(x, y) \\
\text{s.t.} \quad & x \in \mathcal{X}, \quad y \in \mathcal{Y}, \\
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Assumption: Solving over x for fixed y is easy, and the other way around.

Alternating Direction Method

Given: Initial values $(x^0, y^0) \in \mathcal{X} \times \mathcal{Y}$.

for $i = 0, 1, \ldots$ **do**

Choose $x^{i+1} \in \arg\min_{x} \left\{ F(x, y^i) : g(x, y^i) \leq 0, x \in \mathcal{X} \right\}$.

Choose $y^{i+1} \in \arg\min_{y} \left\{ F(x^{i+1}, y) : g(x^{i+1}, y) \leq 0, y \in \mathcal{Y} \right\}$.

end for

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Good (from a theoretical viewpoint): Convergence is well understood

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end for

Good (from a theoretical viewpoint): Convergence is well understood

Issues (from a practical viewpoint):

1. Sub-problems may be infeasible because of the coupling constraints
2. Poor performance

A Penalized Problem

For some penalty parameter $\mu \in \mathbb{R}^m_{>0}$,

$$\begin{aligned} \min_{x,y} \quad & F(x,y) \\ \text{s.t.} \quad & x \in \mathcal{X}, \quad y \in \mathcal{Y} \\ & g(x,y) \leq 0 \end{aligned}$$

$$\downarrow$$

$$\begin{aligned} \min_{x,y} \quad & F(x,y) + \sum_{i=1}^{m} \mu_i [g_i(x,y)]^+ \\ \text{s.t.} \quad & x \in \mathcal{X}, \quad y \in \mathcal{Y} \end{aligned}$$

$[u]^+ = \max\{0, u\}$

Penalty Alternating Direction Method

Given: Initial values $(x^{0,0}, y^{0,0}) \in \mathcal{X} \times \mathcal{Y}$ and $\mu^0 \in \mathbb{R}^r_{\geq 0}$.

for $j = 0, 1, \ldots$ **do**

Set $i \leftarrow 0$

while $(x^{i,j}, y^{i,j})$ is not a partial minimizer of the penalized problem **do**

Choose $x^{i+1} \in \arg\min_x \left\{\phi(x, y^i; \mu) : x \in \mathcal{X}\right\}$

Choose $y^{i+1} \in \arg\min_y \left\{\phi(x^{i+1}, y; \mu) : y \in \mathcal{Y}\right\}$

Set $i \leftarrow i + 1$

end while

Choose new penalty parameters $\mu^{j+1} \geq \mu^j$.

end for

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end for

Good: Convergence is well understood, e.g., Geißler et al. (2017).

Outline

Applying the PADM to the Single-Level Reformulation

Single-Level Reformulation

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Penalized Problem

$$
\begin{aligned}
\min_{x,y,\lambda} \quad & f(x) + \sum_{i=1}^{m} \rho_i \left[b_i - d_{i\cdot}(x)y\right]^+ + \sum_{j=1}^{n} \mu_j \left|c_j - d_{\cdot j}(x)^\top \lambda\right| \\
\text{s.t.} \quad & x \in X, \quad y \in Y, \quad \lambda \geq 0, \quad f^\top y \leq b^\top \lambda
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The Sub-problems

1. Try to find feasible primal-dual point given $\hat{x}$ by solving

$$\min_{y,\lambda} \quad \sum_{i=1}^{m} \rho_i \left[b_i - d_{i\cdot}(\hat{x})y\right]^+ + \sum_{j=1}^{n} \mu_j \left|c_j - d_{\cdot j}(\hat{x})^\top \lambda\right|$$
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$$\text{s.t.} \quad y \in Y, \quad \lambda \geq 0, \quad f^\top y \leq b^\top \lambda$$

2. Try to repair infeasibilities of $(\hat{y}, \hat{\lambda})$ by choosing a new x

$$\min_{x} \quad f(x) + \sum_{i=1}^{m} \rho_i \left[b_i - d_{i\cdot}(x)\hat{y}\right]^+ + \sum_{j=1}^{n} \mu_j \left|c_j - d_{\cdot j}(x)^\top \hat{\lambda}\right|$$

$$\text{s.t.} \quad x \in X$$

Outline

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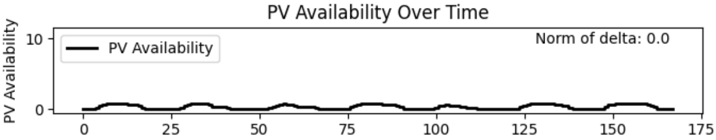
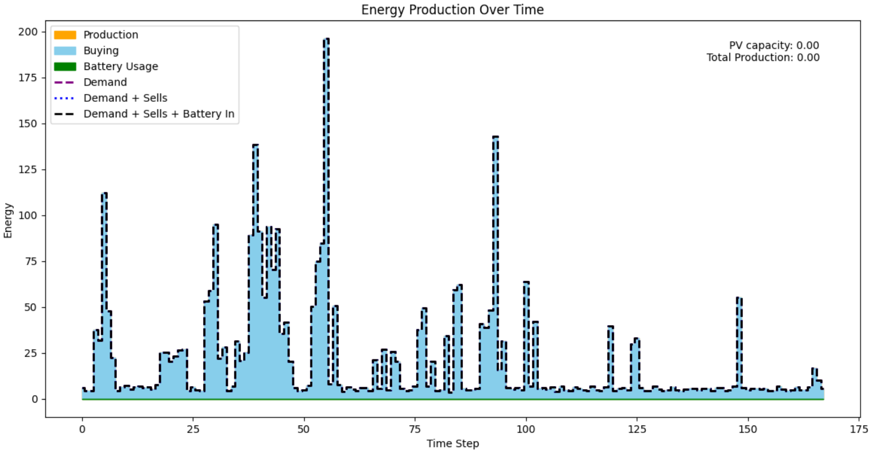
One Example

We consider a given week with

1. costs for buying electricity
2. price for selling electricity surplus
3. house demands
4. cost for buying photovolatic panels
5. historical data on PV availability:



Status Quo: Only Buying From the Grid



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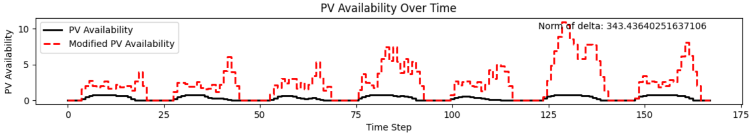
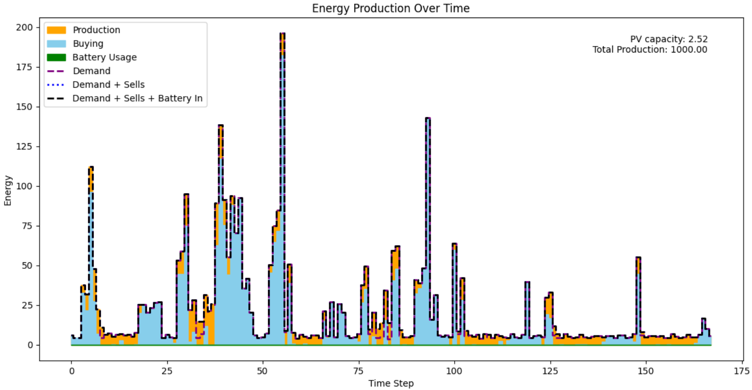
Answer...

Modified Solution





Outline

Setting

Instances based on the NETLIB library, adapted by Kurtz et al. (2024)

Type	Category	Intervals
# Variables	small	$0 \leq n \leq 534$
	medium	$534 \leq n \leq 2167$
	large	$2167 \leq n \leq 22275$
# Constraints	small	$0 \leq m \leq 351$
	medium	$351 \leq m \leq 906$
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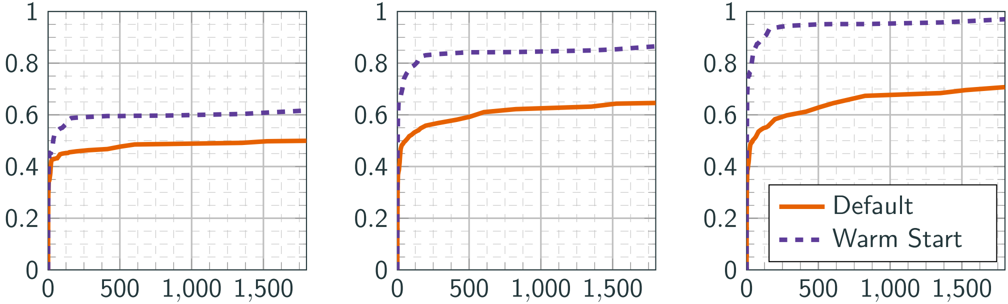
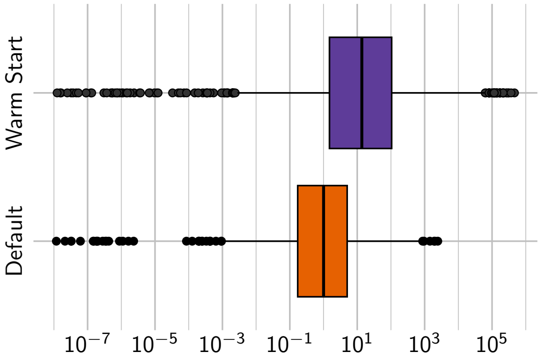
- A total of 5 760 instances
- Not all instances are feasible ($\sim$ 55%)

ECDF of computation times for $f = \|\cdot\|_1$



(a) 1 mutable column **(b)** 5 mutable columns **(c)** 10 mutable columns

Solution Quality



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Conclusion

What We Have Done:

1. Computing counterfactual explanations of linear problems is challenging
2. We derive a single-level reformulation which we heuristically solve by the PADM
3. We can characterize the computed solutions (stationary points of the single-level reformulation)

What to Do Then:

1. Polish the preprint and submit
2. Can we do the same for strong counterfactual explanations
3. Discuss next project steps