Mixed-Integer Linear Bilevel Optimization: A New Single-Level Reformulation

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Problem Statement

- We consider bilevel optimization problems of the form

$$\begin{aligned}\min_{x\in X, y} \quad & c^\top x + d^\top y \\ \text{s.t.} \quad & Ax + By \geq a, \\ & y \in \arg\min_{y' \in Y} \{ f^\top y' \colon Cx + Dy' \geq b \}.\end{aligned}$$

- Using the value function $\varphi(x) := \min_{y\in Y} \{f^\top y \colon Cx + Dy \geq b\}$:

$$\begin{aligned}\min_{x\in X, y\in Y} \quad & c^\top x + d^\top y \\ \text{s.t.} \quad & Ax + By \geq a, \quad Cx + Dy \geq b, \quad f^\top y \leq \varphi(x).\end{aligned}$$

The Follower's Dantzig–Wolfe Reformulation

- Assume $X = \{0,1\}^{n_x}$ and $Y \subset \mathbb{Z}^{n_y - p_y} \times \mathbb{R}^{p_y}$.
- Consider any pair $(\hat{x}, \hat{y})$ such that $\hat{y} \in Y(\hat{x})$, then

$$\varphi(x) = \min_{\alpha \in \{0,1\}^\ell} \left\{ \sum_{k=1}^{\ell} \alpha_k f^\top \hat{y}^k \colon \sum_{k=1}^{\ell} \alpha_k \hat{x}^k = x, \sum_{k=1}^{\ell} \alpha_k = 1 \right\}. \qquad (\star)$$

- One can show that the LP relaxation is tight for all $x \in X$.
- This requires that x is binary.
- One can even show that it is the convex envelope of φ on X.
- With this, we go from the MILP to the LP world!

LP Case: Single-Level Reformulation

- Assume $Y = \mathbb{R}^{n_y}$, then the follower's problem is an LP.
- Replace the follower by its dual problem:

$$\varphi(x) = \max_{\lambda} \{ (b - Cx)^\top \lambda \colon D^\top \lambda = f, \lambda \geq 0 \}.$$

- We obtain the strong-duality single-level reformulation:

$$\begin{aligned}\min_{x\in X, y\in Y, \lambda} \quad & c^\top x + d^\top y \\ \text{s.t.} \quad & Ax + By \geq a, \\ & Cx + Dy \geq b, \\ & f^\top y \leq (b - Cx)^\top \lambda, \\ & D^\top \lambda = f, \ \lambda \geq 0.\end{aligned}$$

- Can be reformulated as a MILP and solved to global optimality.
- See Fortuny-Amat and McCarl (1981).

MILP Case: A New Single-Level Reformulation

- Replace the follower by the dual of the tight LP relaxation of $(\star)$:

$$\varphi(x) = \max_{\pi\in\mathbb{R}, \lambda\in\mathbb{R}^{n_x}} \left\{ \pi + \lambda^\top x \colon \pi + \lambda^\top \hat{x}^k \leq f^\top \hat{y}^k, \ k = 1, \dots, \ell \right\}.$$

- We obtain a new strong-duality single-level reformulation:

$$\begin{aligned}\min_{x\in X, y\in Y, \pi, \lambda} \quad & c^\top x + d^\top y \\ \text{s.t.} \quad & Ax + By \geq a, \\ & Cx + Dy \geq b, \\ & f^\top y \leq \pi + \lambda^\top x, \\ & \pi + \lambda^\top \hat{x}^k \leq f^\top \hat{y}^k, \quad k = 1, \dots, \ell.\end{aligned}$$

- Closed-form bounds on λ and π lead to a MILP.
- Note: exponential number of constraints.
- To be expected since MILP bilevel problems are Σ_2^p-hard.

LP Case: Penalty Alternating Direction Method

- The above single-level reformulation may be too hard to solve.
- We resort to a heuristic approach, the PADM.

with $\rho \nearrow \infty$

$$\begin{aligned}\min_{x\in X, y\in Y} \quad & c^\top x + d^\top y + \rho(b^\top \bar{\lambda} - \bar{\lambda}^\top Cx - f^\top y) \\ \text{s.t.} \quad & Ax + By \geq a, \quad Cx + Dy \geq b.\end{aligned}$$

$(\bar{x}, \bar{y}) \downarrow \quad \uparrow \bar{\lambda}$

$$\begin{aligned}\min_{\lambda} \quad & (b - C\bar{x})^\top \lambda \\ \text{s.t.} \quad & D^\top \lambda = f, \quad \lambda \geq 0.\end{aligned}$$

- See Kleinert and Schmidt (2020), Lefebvre and Schmidt (2024).

MILP Case: Penalty Alternating Direction Method

- The PADM can be extended to MILP bilevel problems.

with $\rho \nearrow \infty$

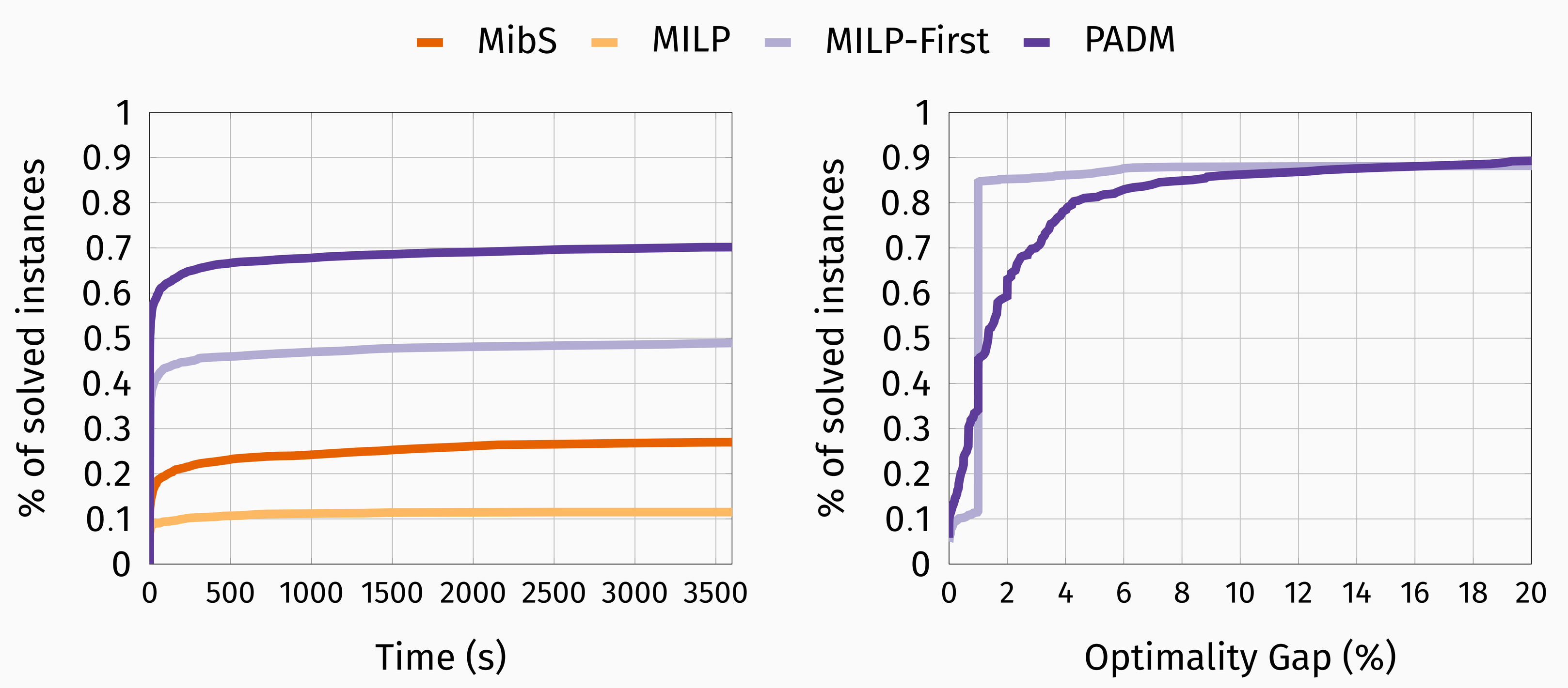
$$\begin{aligned}\min_{x\in X, y\in Y} \quad & c^\top x + d^\top y + \rho(f^\top y - \bar{\pi} - x^\top \bar{\lambda}) \\ \text{s.t.} \quad & Ax + By \geq a, \quad Cx + Dy \geq b,\end{aligned}$$

$(\bar{x}, \bar{y}) \downarrow \quad \uparrow (\bar{\lambda}, \bar{\pi})$

$$\max_{\lambda, \pi} \left\{ \pi + \lambda^\top \bar{x} \colon \pi + \lambda^\top \hat{x}^j \leq f^\top \hat{y}^j, \ j = 1, \dots, \ell \right\}.$$

- The second sub-problem is solved by cut generation.
- It reduces to compute $\varphi(x)$ and $\bar{\lambda} \in \partial \mathrm{vex}_X(\varphi)(\bar{x})$.

Numerical Results on Instances from the BOBILib



Conclusion

- A new single-level reformulation of MILP bilevel problems
  - Can be solved as a MILP via cut generation.
  - Encouraging results.
- Two heuristic approaches
  - Both returning high-quality feasible points.
- Future Work
  - Improve bounds on the dual variables λ and π.
  - Combine this approach with existing techniques like intersection cuts (Fischetti et al., 2018).

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